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EE 364 - Fall 2026

Week 5

Supplemental Notes

Kolmogorov - Foundations Grundbegriffe (1918)

Ω $\longleftrightarrow$ measure space

$(\Omega, \mathcal{A})$ sigma-algebra : abstraction of sample space and assignment of probability to events.

P $\longleftrightarrow$ CAT measure

r.v. X $\longleftrightarrow$ Measurable function (PAM)

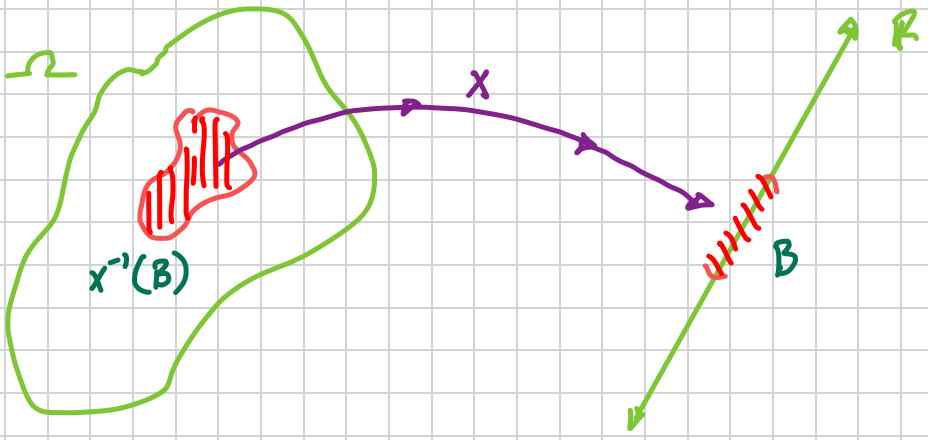
"random experiment"

Defn: $X: \Omega \to \mathbb{R}$ is a random variable (r.v.) on $(\Omega, \mathcal{A})$

iff PAM: Pullbacks are Always Measurable

iff $\forall B \in \mathcal{B}(\mathbb{R})$: $\boxed{X^{-1}(B) \in \mathcal{A}.}$

$\mathcal{B}(\mathbb{R})$: intervals in $\mathbb{R}$.



$\therefore$ $P(X^{-1}(B))$ defined if $X^{-1}(B) \in \mathcal{A}$ since

since $P: \mathcal{A} \longrightarrow [0,1]$

$\therefore$ X is <u>not</u> a random variable if $\exists B \in \mathcal{B}(\mathbb{R})$ w/ $X^{-1}(B) \notin \mathcal{A}$

Ex: flip coin twice $\quad \Omega = \{H,T\} \times \{H,T\}$ $\quad (\mathcal{A} = 2^{\Omega})$

$P(H) = p$

$P(T) = 1-p = q$

Define: X = # heads in 2 flips

$Y = (\#H)^2$

$Z = (\#H)^3 - 1$

} uncountable infinite possible r.v.

Outcomes

	TT	TH	HT	HH
X	0	1	1	2
Y	0	1	1	4
Z	-1	0	0	7

notation doesn't make sense. "function = constant"?

$\therefore$ $P[X=0] = P[X^{-1}(\{0\})] = P[\{TT\}] = q^2$

$P[X=1] = P[X^{-1}(\{1\})] = P[\{TH, HT\}] = 2pq$

$P[X=2] = P[X^{-1}(\{2\})] = P[\{HH\}] = p^2$

$P[X=3] = P[X^{-1}(\{3\})] = P[\emptyset] = 0$

Also: $P[Z=0] = P[Z^{-1}(\{0\})] = P[\{TH, HT\}] = 2pq.$

Ex: (Indicator function) $I_A : \Omega \to \{0,1\}$.

with $A \subset \Omega$ and $A \in \mathcal{Q}$. Define

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A. \\ 0 & \text{if } \omega \notin A. \end{cases}$$

$\therefore I_A^{-1}(\{0\}) = A^c \in \mathcal{Q}$ (by CMT)

$I_A^{-1}(\{1\}) = A \in \mathcal{Q}$

$\therefore I_A$ is a r.v. (is "measureable")

Note: "Fuzzy" set $\longleftrightarrow$ $I_A : \Omega \to [0,1]$ vs. $\{0,1\}$.

Ex: PAM can fail. Define $\Omega = [0,10]$

$\mathcal{Q} = \{\phi, [0,3), [3,10], \Omega\}$.

$X : \Omega \to \Omega$

$X(\omega) = \omega$ if $\omega \in \Omega$ (identity function)

Range S.A. $\mathcal{B}([0,10])$

pick $B = (2,4) \in \mathcal{B}([0,10])$

$\therefore X^{-1}(B) = X^{-1}((2,4)) = (2,4) \notin \mathcal{Q}$.

$\therefore X$ is NOT a r.v. (PAM fails)

Defn: The probability mass function (p.m.f.)

$p_X(x_k) = P[X = x_k]$ where x_k are the outcomes from random experiment with countable sample space

Two properties: (1) $p_X(x_k) \geq 0 \quad \forall x_k \in \Omega$

(2) $\sum_{x_k \in \Omega} p_X(x_k) = 1.$

Ex: binomial "distribution" (random variable)

$P[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$

k successes in n flips

Defn: Cumulative Distribution Function (CDF)

Say $X: \Omega \to \mathbb{R}$ is a r.v. on p.s. $(\Omega, \mathcal{A}, P)$.

Then CDF $F_X(x) \triangleq P(X \leq x)$

$= P(\{\omega \in \Omega : X(\omega) \leq x\})$

$= P(X^{-1}((-\infty, x]))$

Note: CDF F_X always exists.

Random Variable

- X
- function
- foot



Realization

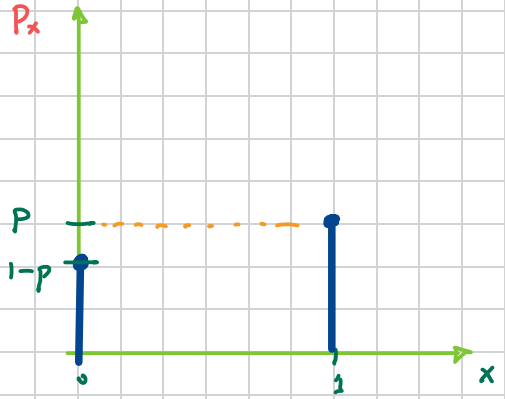
- x
- constant
- footprint.



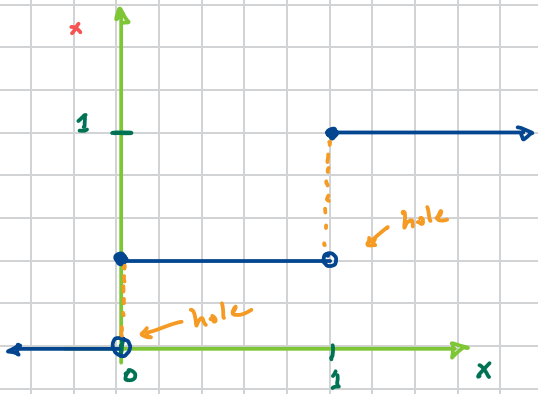
Note: Cumulative distribution function (CDF) for discrete random variable X:

$$F_X(x) = P[X \le x] = \sum_{x_k \in \Omega_X} p_X(x_k)$$

Ex: Bernoulli random variable $P[X=1] = p$, $P[X=0] = 1-p$.



pmf, $p_X(x)$



CDF, $F_X(x)$

Note: pmf is a discrete function, defined only for discrete argument $x_k \in \Omega$. CDF is defined $\forall x \in \mathbb{R}$.

Properties of the CDF:

1. $F_X(-\infty) = 0$
2. $F_X(u_1) \le F_X(u_2)$ if $u_1 \le u_2$ — F_X is increasing
3. $F_X(\infty) = 1$
4. $F_X(b) = \lim_{h \to 0^+} F_X(b+h)$ — F_X is right continuous

4b. $P[X=b] = \lim_{h \to 0^+} F_X(b+h) - \lim_{h \to 0^-} F_X(b+h)$

approach from right; approach from left; "jump"

Working with the CDF:

Consider event $\{X \le u_1\} = \{X \le u_0\} \cup \{u_0 < X \le u_1\}$ for $u_0 < u_1$ (disjoint)

$$\therefore P[\{X \le u_1\}] = P[\{X \le u_0\}] + P[\{u_0 < X \le u_1\}].$$

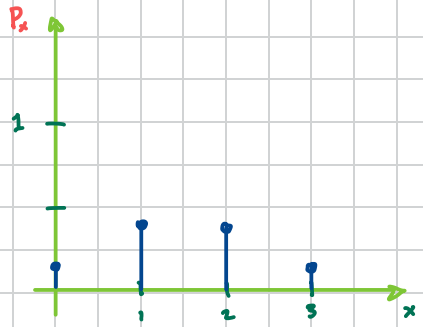
($P[\{X \le u_1\}] = F_X(u_1)$, $P[\{X \le u_0\}] = F_X(u_0)$)

$$\therefore \boxed{P[u_0 < X \le u_1] = F_X(u_1) - F_X(u_0)}$$

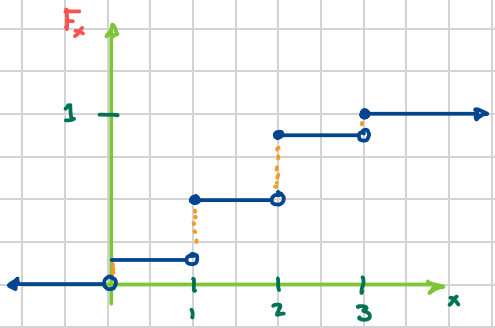
$$\therefore P[u_0 \le X \le u_1] = P[X = u_0] + P[u_0 < X \le u_1]$$

$$= F_X(u_1) - F_X(u_0) + p_X(u_0)$$

Ex: Count # heads in 3 fair coin flips $\quad X \sim b(3, 0.5)$



pmf, $p_X(x)$



CDF, $F_X(x)$

Measuring the size of sets

Defn: $f: X \to Y$ is 1-to-1 ("Injective") iff

$$\forall x \, \forall z : f(x) = f(z) \longrightarrow x = z.$$

Defn: $f: X \to Y$ is ONTO ("Surjective") iff

$$\forall y \in Y \; \exists x \in X : y = f(x) \qquad (f(X) = Y)$$

Defn: $f: X \to Y$ is a BIJECTION ("1-to-1 correspondence")

iff f is 1-to-1 and onto.

$|A|$ = cardinality ("size") of set A.

$\mathbb{N} = \{1, 2, 3, \ldots\}$ natural numbers

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ integers

$\mathbb{Z}^+ = \mathbb{N}$

$\mathbb{R} = (-\infty, \infty)$ reals

Defn: A is finite iff $A \overset{\text{1-to-1, onto}}{\longleftrightarrow} S$ for some $S \subset \mathbb{N}$

such that $A = \{a_1, \ldots, a_n\}$ for some $n \in \mathbb{N}$ $(\text{or } A = \emptyset)$

Else A is infinite.

Fact: A infinite $\longleftrightarrow$ $A \overset{\text{1-to-1, onto}}{\longleftrightarrow} B$ for some $B \subset A$ and $B \neq A$.

$\therefore$ infinite $\longleftrightarrow$ $\sim$ finite

Cantor's theorem: $|X| < |2^X|$

Defn: A is denumerable iff $A \overset{\text{1-to-1, onto}}{\longleftrightarrow} \mathbb{N}$ (e.g. $\mathbb{Z}$)

Defn: A is countable iff (1) A is finite, or

(2) A is denumerable

Facts: $|\mathbb{Z}| = |\mathbb{N}| = \aleph_0$ (aleph-nought)

$|\mathbb{R}| = |2^{\mathbb{Z}}| = c = \aleph_1$

↑ power of the continuum

$|A| = \aleph_k \Longrightarrow |2^A| = \aleph_{k+1}$ $(k = 0, 1, \ldots)$

Cantor's Continuum Hypothesis There is no ω such that

$\aleph_k < \omega < \aleph_{k+1}$

Ex: $(0,1)$ same size as $\mathbb{R}^+$

$f(x) = \frac{1}{1 + e^{-x}}$ (note: adding 2-points $0, 1 \longrightarrow [0,1]$

no 1-to-1 onto map w/ $\mathbb{R}^+$)

Aside: $\pm\infty$ not real numbers

Continuous probability distributions ("densities")

Suppose sample space Ω not countable → ∴ CDF continuous.

e.g., $\Omega = [0,1]$

$\Omega = \mathbb{R}$

Defn: $f: \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$

iff $\forall \varepsilon > 0 \; \exists \delta > 0 : |x - x_0| < \delta \longrightarrow |f(x) - f(x_0)| < \varepsilon.$

inputs x, x_0 close $\longrightarrow$ outputs $f(x), f(x_0)$ close

- f is continuous at a (∴ left + right continuous)
  - reminder: CDF only requires right continuous
  - $\longleftrightarrow \lim_{x \to a} f(x) = f(a)$ ↑ if f defined at a
  - $\longleftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 : |x - a| < \delta \longrightarrow |f(x) - f(a)| < \varepsilon.$

Defn: f is continuous iff it is continuous at $\forall x_0 \in \mathbb{R}$.

Fact: $P[X = b] = \lim_{h \to 0^+} F_X(b+h) - \lim_{h \to 0^-} F_X(b+h)$

(right limit; left limit)

$= 0$ if F_X continuous

↑ not "close to zero" or "very small", identically 0

∴ If X is a continuous random variable (∴ CDF continuous)

then $P[X = x] = 0 \quad \forall x \in \mathbb{R}.$

i.e. $P[X = a] = P[X = b] = 0$

$$\therefore \boxed{P[a < X \le b] = P[a \le X \le b] = P[a \le X < b] = P[a < X < b]}$$

Q: If $P[X = x] = 0$ for all x how to describe probability of an outcome? (e.g., $P[\text{height} = 6\text{ ft}] = 0$)

A: Use density of probability "near" the outcome (e.g., $P[\text{height} \approx 6\text{ ft}]$)

*

Thm: CDF F is absolutely continuous iff $\exists$ probability density function (pdf) f:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt. \quad \therefore \; ① \; \boxed{f = F' = \frac{dF}{dx}}$$

$$② \int_{-\infty}^{\infty} f(x)\,dx = 1$$

Defn: The probability density function (pdf) of a continuous random variable X: $\boxed{f_x(x) = \frac{dF_x}{dx}}$ (if it exists)

iff X is "absolutely continuous"

$$\therefore \; F_x(x) = \int_{-\infty}^{x} f_x(t)\,dt.$$

Fundamental theorem of calculus.

$$\int_a^b f(x)\,dx = \underbrace{F(b) - F(a)}_{\text{"anti-derivative"}}$$

i.e. $P[a \le X \le b] = F_x(b) - F_x(a) = \int_a^b f_x(x)\,dx$ (if it exists.)

- behavior of interior is completely determined by behavior at boundary.
- simplifies operation of probability because it hides underlying complexity.

Differentiable functions

Defn: function f is differentiable at a if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a) \quad \text{exists.}$$

Ex: $f(x) = |x|$ continuous everywhere

not differentiable (at $x = 0$)

Thm: Differentiable $\longrightarrow$ Continuous

Pf: $f(x) = f(x) + f(a) - f(a) = \left(\frac{f(x) - f(a)}{x - a} \right)(x - a) + f(a)$

$$\therefore \lim_{x \to a} f(x) = \lim_{x \to a} \left[\left(\frac{f(x) - f(a)}{x - a} \right)(x - a) + f(a) \right]$$

$$= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right) \cdot \lim_{x \to a} (x - a) + f(a)$$

$\lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right) = f'(a)$ exists by assumption

$\lim_{x \to a} (x - a) = 0$

$$= f(a)$$

$$\therefore \lim_{x \to a} f(x) = f(a)$$

$\therefore$ f continuous at a.

$\therefore$ f continuous (since $\forall a$)

QED.

Insight: The pdf specifies the density of probability.

Ex: $P[x < X \leq x + \Delta x] = F_x(x+\Delta x) - F_x(x)$

$$= \Delta x \, \frac{F_x(x+\Delta x) - F_x(x)}{\Delta x}$$

$\therefore$ for small h. $\simeq f_x(x) \cdot \Delta x$ $\Longrightarrow$ (density) = (prob / length) · length

($f_x(x)$ is the derivative of $F_x(x)$)

So $f_x(x)$ gives the probability that X falls in a small interval (width Δx) around x.

Properties of the pdf

1. $f_x(x) \geq 0$
2. $P[a \leq X \leq b] = \int_a^b f_x(x)\,dx$ — "area under the curve"
3. $F_x(x) = \int_{-\infty}^{x} f_x(t)\,dt$.
4. $\int_{-\infty}^{\infty} f_x(t)\,dt = 1$ — "normalization condition"

Note: (by 4) Given any non-negative function (piecewise continuous) $g(x)$ with $\int_{-\infty}^{\infty} g(x)\,dx = c < \infty$ then $f_x(x) = \frac{g(x)}{c}$ is a valid pdf

BEG CUP, important EE364 pdfs.

B : binomial (and discrete sampling)

E : exponential (gamma) — "waiting time"

G : Gaussian / normal — "thin-tailed" bell

C : Cauchy — "thick-tailed" bell.

U : Uniform

P : Poisson

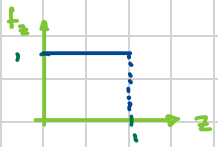
BEG CUP → Uniform pdf

$X \sim U[a,b]$: $f_x(x) = \frac{1}{b-a}$ if $x \in [a,b]$, else $= 0$.

$\therefore$ CDF is a "ramp"



Standard Uniform $Z \sim U(0,1)$



Ex: $X \sim U[0,10]$

$$\therefore f_x(x) = \begin{cases} 1/10 & 0 \le x \le 10 \\ 0 & \text{else} \end{cases}$$

$$P[X \le 5] = \int_{-\infty}^{5} f_x(x)\,dx = F_x(5)$$

$$= \int_0^5 \frac{1}{10}\,dx = \frac{1}{2}$$

$$F_x(x) = \begin{cases} 0 & x < 0 \\ 1/10\, x & 0 \le x < 10 \\ 1 & x \ge 10 \end{cases}$$

$$P[2 \le X \le 6] = \int_2^6 \frac{1}{10}\,dx = \frac{4}{10} \quad (= 2/5)$$

$F_x(6) - F_x(2)$

Later: Beta random variable.

BEG CUP → Gaussian pdf (Normal pdf)

$X \sim N(\mu, \sigma^2)$: $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

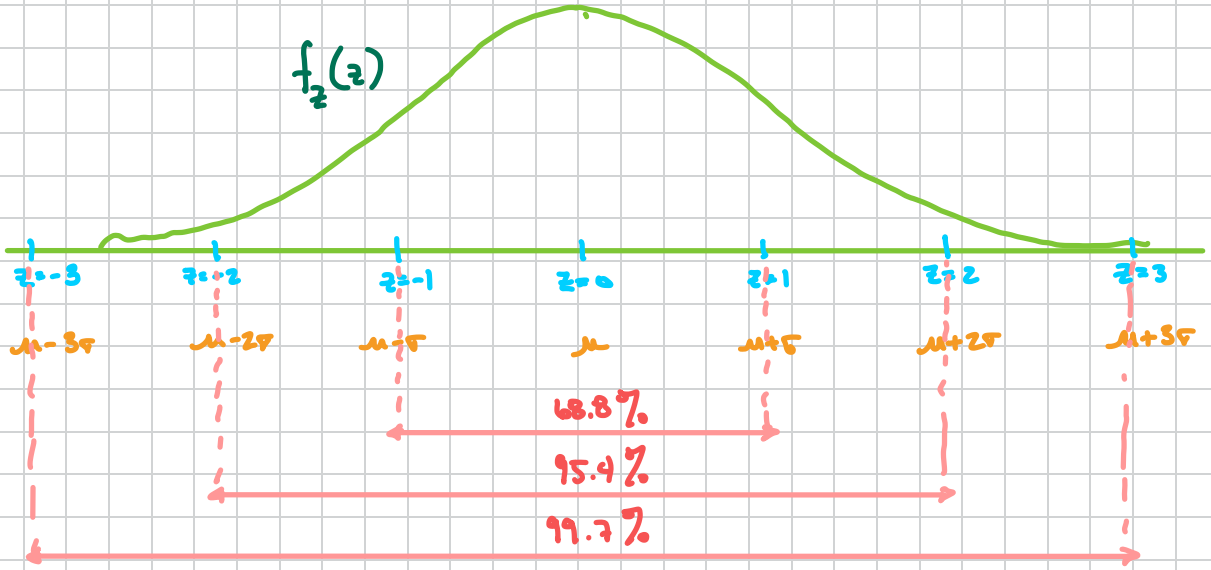
$\mu \in \mathbb{R}$, $\sigma > 0$, $x \in \mathbb{R}$

μ: mean, σ^2: variance, σ: standard deviation

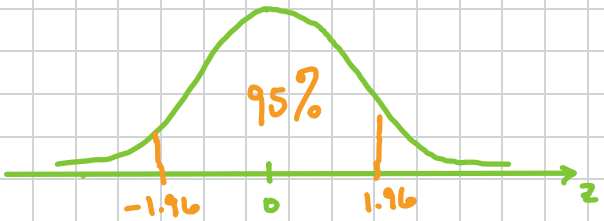
first → **Standard Normal:**

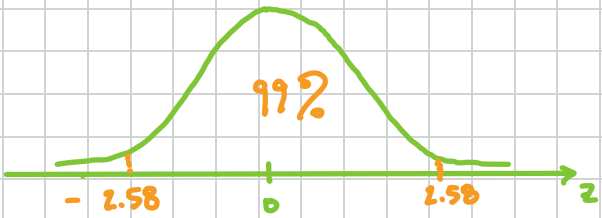
$Z \sim N(0,1)$: $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$Z \triangleq \frac{X-\mu}{\sigma}$ standardize



related to sums of random variables





95% ↔ $z = \pm 1.96$ ($\approx \pm 2\sigma$)

99% ↔ $z = \pm 2.58$

Fact: Gaussian CDF does not have a closed form (but CDF does exist)

$$F_Z(z) = \Phi(z) = \int_{-\infty}^{z} f_t(t)\,dt = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt.$$

"Phi-function"

standardize

$$f_X(x) = \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right)$$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{x} \frac{1}{\sigma} f_Z\left(\frac{t-\mu}{\sigma}\right) dt$$

$$= \int_{-\infty}^{\frac{x-\mu}{\sigma}} f_Z(z)\,dz$$

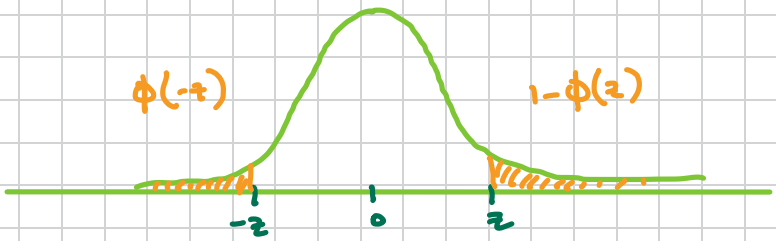
$$= \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Ex: Suppose $X \sim N(\mu, \sigma^2)$

$$P[a < X \le b] = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Note: $\Phi(-z) = 1 - \Phi(z)$ (symmetry)

$P[Z \le -z] = P[Z \ge z]$



Advanced result: $\lim_{\sigma \to 0} \frac{1}{\sigma} f_Z\left(\frac{x_0-\mu}{\sigma}\right) = \delta(x_0 - \mu)$

Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Prf:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)} = \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy}$$

polar

$$= \sqrt{\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=\infty} r\,e^{-r^2}\,dr\,d\theta} \qquad dx\,dy \longmapsto r\cdot dr\,d\theta$$

$$= \sqrt{2\pi}\cdot\sqrt{\int_0^{\infty} r\cdot e^{-r^2}\,dr} = \sqrt{2\pi}\cdot\sqrt{\tfrac{1}{2}} = \sqrt{\pi}$$

QED.

Normal Table Values

$X \sim N(\mu, \sigma^2)$ CDF: $F_x(x) = \int_{-\infty}^{x} f(w)\,dw$

$= \frac{1}{\sqrt{2\pi}\sigma} \cdot \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{w-\mu}{\sigma}\right)^2} dw$

no closed form

$Z \sim N(0,1)$ $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-w^2/2}\,dw$

$\therefore$ use table

Ex: $X \sim N(1,4)$

$P[X \le 3] = F_x(3) = F_z\left(\frac{3-1}{2}\right) = \Phi(1)$

$\frac{X-\mu}{\sigma} \le \frac{3-\mu}{\sigma}$ $\quad Z$

$P[X \le -1] = P\left[\overset{Z}{\frac{X-1}{2}} \le \frac{-1-1}{2}\right] = \Phi(-1)$

$P[-1 \le X \le 5] = P\left[\frac{-1-1}{2} \le \overset{Z}{\frac{X-1}{2}} \le \frac{5-1}{2}\right] = \Phi(2) - \Phi(-1)$

$P[X \ge 2] = 1 - P[X \le -2] = 1 - P\left[\overset{Z}{\frac{X-1}{2}} \le \frac{-2-1}{2}\right] = \Phi(-1.5)$

Note: Q-function

$Q(x) = 1 - \Phi(x) = P(Z > z)$

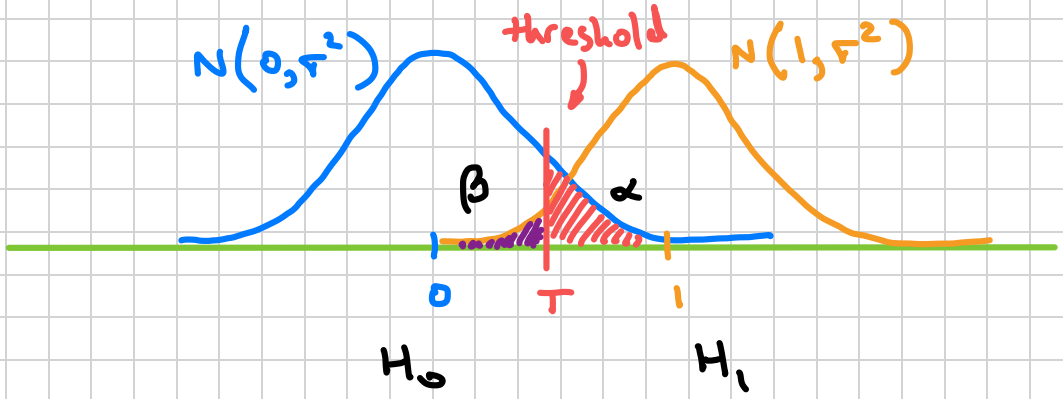
$= \frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)$

for "error function" $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-w^2}\,dw$

Table 1: Table of the Standard Normal Cumulative Distribution Function $\Phi(z)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Signal Detection



H_0: no signal $\quad X \sim N(0,\sigma^2)$

H_a: signal present $\quad X \sim N(1,\sigma^2)$

decision rule

H_0 if $X \leq T$

H_1 if $X > T$

$$\alpha \triangleq P[\text{Type 1 error}] = P[\text{reject } H_0 \mid H_0 \text{ true}].$$

$$= P[X > T \mid H_0]$$

$$= P[\text{false alarm}] \quad \text{false positive}$$

$$\therefore 1 - \alpha = P[\text{correct rejection}].$$

$$\beta \triangleq P[\text{Type 2 error}] = P[\text{accept } H_0 \mid H_a \text{ true}].$$

$$= P[X \leq T \mid H_a]$$

$P[X>T|H_a]$

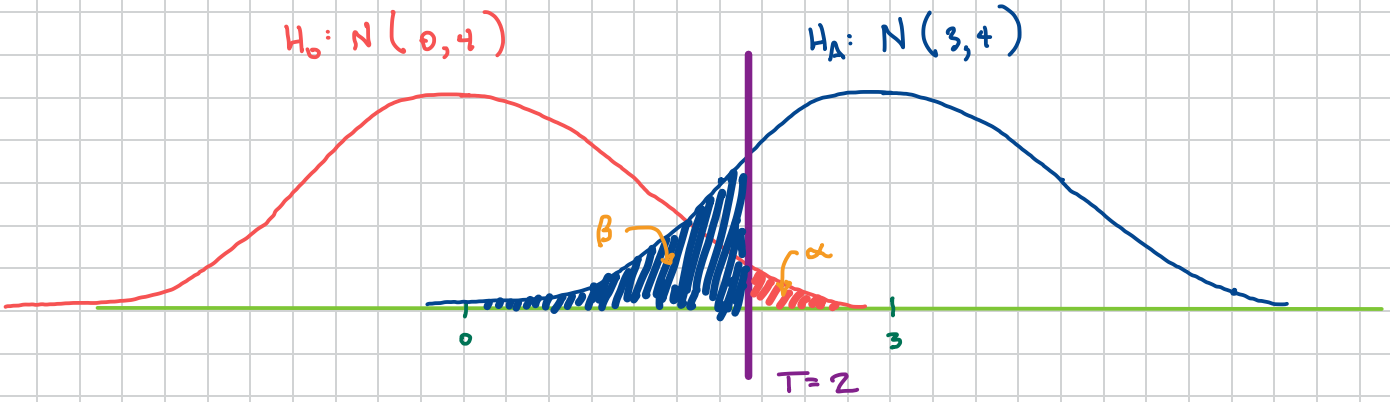
$$= P[\text{miss}] \quad \text{false negative}$$

$$\therefore \text{"Power"} \triangleq 1 - \beta = P[\text{Hit}] = P[\text{Correct detection}] = P_D$$

$\therefore \alpha \downarrow \longleftrightarrow \beta \uparrow$

neyman pearson detection:
maximize P_D for fixed false alarm
e.g. $\alpha = 0.05$.

Ex:



Threshold $T = 2$

$$\alpha = P(\text{type 1 error}) = P(\underbrace{\text{reject } H_0}_{X>2} \mid \underbrace{H_0 \text{ true}}_{N(0,4)})$$

$$= P(X \geq 2 \mid X \sim N(0,4))$$

$$= P\left(\underbrace{\frac{X-0}{2}}_{Z \sim N(0,1)} \geq \frac{2-0}{2}\right)$$

$$= P(Z \geq 1) = 1 - 0.9332 = \boxed{0.06668}$$

$$\beta = P(\text{type 2 error}) = P(\underbrace{\text{accept } H_0}_{X<2} \mid \underbrace{H_A \text{ true}}_{N(3,4)})$$

$$= P\left(\underbrace{\frac{X-3}{2}}_{Z \sim N(0,1)} \leq \frac{2-3}{2}\right)$$

$$= P\left(Z \leq -\frac{1}{2}\right) = \boxed{0.3085}$$

Ex: (at home) repeat, $T = 1$ (move threshold, left $2 \to 1$)

$\alpha \uparrow$, $\beta \downarrow$ (tradeoff)

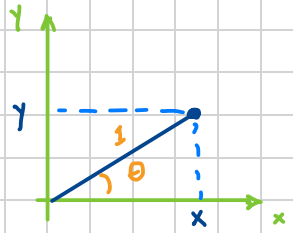
Ex: (at home) find T so that $\alpha = \beta$.

Trigonometry Review

Pythagoreas: $c^2 = a^2 + b^2$ (right triangle)

Cosine Law: $c^2 = a^2 + b^2 - 2ab \cdot \cos\theta$

(Pythag. $\cos\frac{\pi}{2} = 0$)



$\cos\theta = \frac{x}{r}$

$\sin\theta = \frac{y}{r}$

$\therefore x^2 + y^2 = r^2 \cdot \cos^2\theta + r^2 \cdot \sin^2\theta = r^2 \cdot (\sin^2\theta + \cos^2\theta)$

$\therefore r = 1 \longrightarrow \sin^2\theta + \cos^2\theta = 1$

$(x^2 + y^2 = 1)$

$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad (= \frac{y}{x})$

$1 + \tan^2\theta = 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta$

$\frac{d(\tan\theta)}{d\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = 1 + \tan^2\theta = \sec^2\theta$

Taylors Series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\therefore e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \frac{i^n\theta^n}{n!} = 1 + \frac{i^0\theta^0}{0!} + \frac{i^1\theta^1}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \cdots + \frac{i^n\theta^n}{n!} + \cdots$

$= \left(\frac{\theta^0}{0!} - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \cdots\right)$

$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot \theta^{2n}}{(2n)!} = \cos\theta \qquad \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} = \sin\theta.$

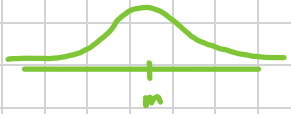
$= \cos\theta + i \cdot \sin\theta.$

Special case: set $\theta = \pi$ $\therefore \cos\theta = -1$, $\sin\theta = 0$

$$\therefore e^{i\pi} + 1 = 0$$

BEG CAP → Cauchy pdf

"Thick tail" bell curve
(infinite variance)



$X \sim C(m, d)$: $f_x(x) = \frac{1}{\pi d\left(1 + \left(\frac{x-m}{d}\right)^2\right)}$

scale or "dispersion" $d > 0$ location: m

CDF: $F = \tan^{-1} + \frac{1}{2}$

$$F_x(x) = P(X \le x) = \frac{1}{\pi} \tan^{-1}\left(\frac{x-m}{d}\right) + \frac{1}{2}$$

Standard Cauchy:

$Z \sim C(0,1)$: $f_z(z) = \frac{1}{\pi(1+z^2)}$

SαS : Symmetric Alpha-Stable pdf.

family of bell-curves: $0 < \alpha \le 2$

$\alpha \downarrow \longrightarrow$ thicker tails

$\alpha = 2$: Gaussian

$\alpha = 1$: Cauchy

(related: Generalized Central Limit Theorem)

Thm: $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$

Pf: $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ Let $x = \tan\theta$

$dx = d(\tan\theta) = (1+\tan^2\theta)\,d\theta$

$\therefore = \int_{-\pi/2}^{\pi/2} \frac{1+\tan^2\theta}{1+\tan^2\theta}\,d\theta = \int_{-\pi/2}^{\pi/2} 1\,d\theta = \theta\Big|_{\theta=-\pi/2}^{\theta=\pi/2} = \pi.$

QED.

$\therefore Z \sim \text{Cauchy}(0,1)$ $\quad f_z(z) = \frac{1}{\pi(1+z^2)}$

$\therefore \int_{-\infty}^{\infty} f_z(z)\,dz = \int_{-\infty}^{\infty} \frac{1}{\pi(1+z^2)}\,dz = 1.$

Defn: Gamma function $\Gamma(\alpha)$ for $\alpha > 0$.

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$$

($x^{\alpha-1}$: power function; e^{-x}: exponential function)

Integration by parts

Thm: $\int u\, dv = uv - \int v\, du$ $\quad \left(\int u(x)\, dv(x) = u(x)\cdot v(x) - \int v(x)\, du(x) \right)$

("inverse" chain rule.)

Pf: $(uv)' = u'v + uv' \simeq du\, v + u\, dv$

$\therefore u\, dv = (uv)' - v\, du$

$\therefore \int u\, dv = uv - \int v\, du.$

QED.

Thm: $\Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)$

Pf: $\Gamma(\alpha+1) = \int_0^\infty x^{\alpha+1-1} e^{-x}\, dx$

Let $u = x^\alpha$, $du = \alpha x^{\alpha-1} dx$; $dv = e^{-x} dx$, $v = -e^{-x}$

$$= -x^\alpha (e^{-x}) \Big|_{x=0}^{x=\infty} - \int_0^\infty (-e^{-x})(\alpha x^{\alpha-1}\, dx)$$

$$= -(0-0) + \alpha \underbrace{\int_0^\infty x^{\alpha-1} e^{-x}\, dx}_{\Gamma(\alpha)}$$

$$= \alpha \cdot \Gamma(\alpha)$$

QED.

$\therefore \Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)$ if $\alpha \in \mathbb{R}^+$

★ $\therefore \Gamma(n+1) = n!$ if $n \in \mathbb{Z}^+$.

Thm: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Pf:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{\frac{1}{2}-1} e^{-x}\, dx$$

$$= \int_0^\infty x^{-\frac{1}{2}} e^{-x}\, dx$$

Let $u = \sqrt{x} = x^{1/2}$

$du = \frac{1}{2} x^{-1/2}\, dx$

$$= \int_{u=0}^{u=\infty} 2 e^{-u^2}\, du$$

$$= \int_{-\infty}^{+\infty} e^{-u^2}\, du.$$

(Gaussian integral)

$$= \sqrt{\pi}$$

BEG CUP → Exponential pdf.

$X \sim \exp(\theta)$: $f(x) = \frac{1}{\theta} \cdot e^{-x/\theta}$ if $x > 0$, $\theta > 0$.

BEG CUP
- γ < χ^2 < t, F
- Erlang
- Extreme Value pdfs (EV).
  - Weibull
  - Frechet
  - Gumbel

} $\max(X_1, \ldots, X_n)$

E.V. theorem (Fisher-Tippets)

Gamma pdf:

$X \sim \gamma(\alpha, \theta)$ $\quad x > 0, \alpha > 0, \theta > 0$

$f_x(x) = \frac{x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha) \cdot \theta^{\alpha}}$

$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} \, dx.$

∴ 3-special cases

① $\alpha = 1$ ⟶ Exponential ("waiting time") next, k=1

② $\alpha \in \mathbb{Z}^+$ ⟶ Erlang — k waiting time

③ $\alpha = \frac{r}{2}, \theta = 2$ ⟶ $X \sim \chi^2(r)$

$r \in \mathbb{Z}^+$ — for "r degrees of freedom"

Recall: $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy$

put $y = \frac{x}{\theta}$ $(\theta > 0)$ $\therefore$ $dy = \frac{1}{\theta} \cdot dx$

$$\Gamma(\alpha) = \int_{x=0}^{x=\infty} e^{-x/\theta} \cdot \left(\frac{x}{\theta}\right)^{\alpha-1} \cdot \frac{1}{\theta}\, dx$$

$$= \frac{1}{\theta^\alpha} \cdot \int_0^\infty x^{\alpha-1} e^{-x/\theta}\, dx.$$

$\therefore$ $X \sim \gamma(\alpha, \theta)$ $\quad f_x(x) = \dfrac{x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha)\, \theta^\alpha}$, $\alpha > 0$, $\theta > 0$, $x \geq 0$

$\therefore$ $\displaystyle\int_0^\infty \frac{x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha) \cdot \theta^\alpha}\, dx = \frac{1}{\Gamma(\alpha)} \cdot \underbrace{\int_0^\infty \frac{x^{\alpha-1} e^{-x/\theta}}{\theta^\alpha}\, dx}_{\Gamma(\alpha)} = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1$

$\therefore$ $f_x(x)$ is a valid pdf.

Chi-Square pdf

$X \sim \gamma\left(\frac{r}{2}, 2\right) = \chi^2(r)$ with "r-degrees of freedom"

$$f(x) \triangleq \frac{x^{\frac{r}{2}-1} e^{-x/2}}{\Gamma(r/2)\, 2^{r/2}} \quad \text{if } x > 0 \quad (r > 0)$$

$\therefore X \sim \gamma(\alpha, \theta)$ if $\alpha = \frac{r}{2}$ & $\theta = 2$.

Sums (3) $\sum_{k=1}^{n} X_k \sim \chi^2\left(\sum_{k=1}^{n} r_k\right)$ if independent $X_k \sim \chi^2(r_k)$

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r-even: $\Gamma\left(\frac{r}{2}\right) = \left(\frac{r}{2}-1\right)!$ r-odd: $\Gamma\left(r+\frac{1}{2}\right) = \frac{(2r)!}{r!\,4^r}\sqrt{\pi}$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ ← Gaussian integral

$\Gamma(1) = 1$ mass of Exp(1) pdf

$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$ from $\Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)$

$\Gamma(2) = 1$

$\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$

$\Gamma(3) = 2$

$\Gamma\left(\frac{7}{2}\right) = \frac{15}{8}\sqrt{\pi}$

$\Gamma(4) = 3! = 6$

Ex: Find $P[3.25 \leq X \leq 20.5]$ if $X \sim \chi^2(10)$

Using R: $P[3.25 \leq X \leq 20.5] = P[X \leq 20.5] - P[X \leq 3.25]$

$$= \text{pchisq}(20.5, 10) - \text{pchisq}(3.25, 10)$$

$$= 0.9751371 - 0.0250865$$

$$= 0.9500506$$

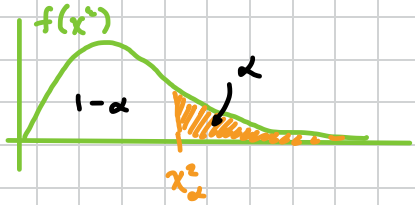
Using table:

$\chi^2_\alpha(10)$ closest to 3.25

$P[X \leq 3.25] \approx 1 - \alpha = 1 - 0.975 = 0.025$

$P[X \leq 20.5] \approx 1 - \alpha = 1 - 0.025 = 0.975$

$\chi^2_\alpha(10)$ closest to 20.5



$\therefore P[3.25 \leq X \leq 20.5] = P[X \leq 20.5] - P[X \leq 3.25]$

$$= 0.975 - 0.025$$

$$= 0.95.$$

Degrees of Freedom	$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$
1	2.70554	3.84146	5.02389	6.63490	7.87944
2	4.60517	5.99147	7.37776	9.21034	10.5966
3	6.25139	7.81473	9.34840	11.3449	12.8381
4	7.77944	9.48773	11.1433	13.2767	14.8602
5	9.23635	11.0705	12.8325	15.0863	16.7496
6	10.6446	12.5916	14.4494	16.8119	18.5476
7	12.0170	14.0671	16.0128	18.4753	20.2777
8	13.3616	15.5073	17.5346	20.0902	21.9550
9	14.6837	16.9190	19.0228	21.6660	23.5893
10	15.9871	18.3070	20.4831	23.2093	25.1882
11	17.2750	19.6751	21.9200	24.7250	26.7569
12	18.5494	21.0261	23.3367	26.2170	28.2995
13	19.8119	22.3621	24.7356	27.6883	29.8194
14	21.0642	23.6848	26.1190	29.1413	31.3193
15	22.3072	24.9958	27.4884	30.5779	32.8013
16	23.5418	26.2962	28.8454	31.9999	34.2672
17	24.7690	27.5871	30.1910	33.4087	35.7185
18	25.9894	28.8693	31.5264	34.8053	37.1564
19	27.2036	30.1435	32.8523	36.1908	38.5822
20	28.4120	31.4104	34.1696	37.5662	39.9968
21	29.6151	32.6705	35.4789	38.9321	41.4010
22	30.8133	33.9244	36.7807	40.2894	42.7956
23	32.0069	35.1725	38.0757	41.6384	44.1813
24	33.1963	36.4151	39.3641	42.9798	45.5585
25	34.3816	37.6525	40.6465	44.3141	46.9278
26	35.5631	38.8852	41.9232	45.6417	48.2899
27	36.7412	40.1133	43.1944	46.9630	49.6449
28	37.9159	41.3372	44.4607	48.2782	50.9933
29	39.0875	42.5569	45.7222	49.5879	52.3356
30	40.2560	43.7729	46.9792	50.8922	53.6720
40	51.8050	55.7585	59.3417	63.6907	66.7659
50	63.1671	67.5048	71.4202	76.1539	79.4900
60	74.3970	79.0819	83.2976	88.3794	91.9517
70	85.5271	90.5312	95.0231	100.425	104.215
80	96.5782	101.879	106.629	112.329	116.321
90	107.565	113.145	118.136	124.116	128.299
100	118.498	124.342	129.561	135.807	140.169

Source: From Thompson, C. M. "Tables of the percentage points of the χ^2-distribution." *Biometrika*, 1941, Vol. 32, pp. 188–189. Reproduced by permisson of the *Biometrika* Trustees.

$r = 10$

$P(X \leq 20.5) \approx 1 - \alpha$

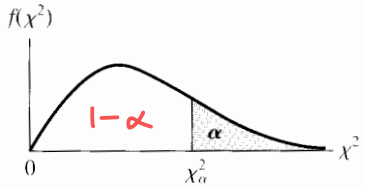
$= 1 - 0.025$

$= 0.975$

Exact:

$P(X \leq 20.5) = 0.9751371$

TABLE 8 Critical Values of χ^2



Degrees of Freedom	$\chi^2_{.995}$	$\chi^2_{.990}$	$\chi^2_{.975}$	$\chi^2_{.950}$	$\chi^2_{.900}$
1	.0000393	.0001571	.0009821	.0039321	.0157908
2	.0100251	.0201007	.0506356	.102587	.210720
3	.0717212	.114832	.215795	.351846	.584375
4	.206990	.297110	.484419	.710721	1.063623
5	.411740	.554300	.831211	1.145476	1.61031
6	.675727	0.872085	1.237347	1.63539	2.20413
7	.989265	1.239043	1.68987	2.16735	2.83311
8	1.344419	1.646482	2.17973	2.73264	3.48954
9	1.734926	2.087912	2.70039	3.32511	4.16816
10	2.15585	2.55821	3.24697	3.94030	4.86518
11	2.60321	3.05347	3.81575	4.57481	5.57779
12	3.07382	3.57056	4.40379	5.22603	6.30380
13	3.56503	4.10691	5.00874	5.89186	7.04150
14	4.07468	4.66043	5.62872	6.57063	7.78953
15	4.60094	5.22935	6.26214	7.26094	8.54675
16	5.14224	5.81221	6.90766	7.96164	9.31223
17	5.69724	6.40776	7.56418	8.67176	10.0852
18	6.26481	7.01491	8.23075	9.39046	10.8649
19	6.84398	7.63273	8.90655	10.1170	11.6509
20	7.43386	8.26040	9.59083	10.8508	12.4426
21	8.03366	8.89720	10.28293	11.5913	13.2396
22	8.64272	9.54249	10.9823	12.3380	14.0415
23	9.26042	10.19567	11.6885	13.0905	14.8479
24	9.88623	10.8564	12.4011	13.8484	15.6587
25	10.5197	11.5240	13.1197	14.6114	16.4734
26	11.1603	12.1981	13.8439	15.3791	17.2919
27	11.8076	12.8786	14.5733	16.1513	18.1138
28	12.4613	13.5648	15.3079	16.9279	18.9392
29	13.1211	14.2565	16.0471	17.7083	19.7677
30	13.7867	14.9535	16.7908	18.4926	20.5992
40	20.7065	22.1643	24.4331	26.5093	29.0505
50	27.9907	29.7067	32.3574	34.7642	37.6886
60	35.5346	37.4848	40.4817	43.1879	46.4589
70	43.2752	45.4418	48.7576	51.7393	55.3290
80	51.1720	53.5400	57.1532	60.3915	64.2778
90	59.1963	61.7541	65.6466	69.1260	73.2912
100	67.3276	70.0648	74.2219	77.9295	82.3581

$r = 10$

$P(X \le 3.25) = 1-\alpha$

$= 1-0.975$

$= 0.025$

Exact:

$P(X < 3.25) = 0.250865$

$X \sim \chi^2(r)$ — r d.f.

$P[X \le x] \approx 1-\alpha$ for χ^2_α closest to x.

Beta random variable

$X \sim \text{Beta}(\alpha, \beta)$: $f_X(x) = \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$

if $0 < x < 1$ (else $= 0$)

$\therefore$ $\boxed{\alpha = \beta = 1 \longrightarrow X \sim U[0,1].}$

(n-dim, Dirichlet pdf)

Defn: **Beta function** $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$ $\alpha > 0$, $\beta > 0$

$= \dfrac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ (see below)

★ **Thm:** $B(\alpha,\beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,du \qquad \alpha>0,\ \beta>0$

$$= \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Pf: $$\Gamma(\alpha)\cdot\Gamma(\beta) = \int_{x=0}^{x=\infty} e^{-x}x^{\alpha-1}\,dx \int_{y=0}^{y=\infty} e^{-y}y^{\beta-1}\,dy$$

Fubini $$= \int_{y=0}^{y=\infty}\left[\int_{x=0}^{x=\infty} e^{-(x+y)}x^{\alpha-1}y^{\beta-1}\,dx\right]dy$$

$$= \int_{y=0}^{y=\infty}\int_{x=0}^{x=\infty} f\big(x(u,v),\,y(u,v)\big)\,dx\,dy$$

Let: $x = x(u,v) = uv \qquad y = y(u,v) = u(1-v)$ double substitution

Given: $0<x<\infty$ and $0<y<\infty$

$\therefore\ 0<x+y = u\cdot v + u(1-v) = \cancel{uv} + u - \cancel{uv} = u$

$\therefore\ u>0$

$\therefore\ x>0 \longrightarrow uv>0$

$\therefore\ v>0$ since $u>0$

$\therefore\ u<\infty$ since $x = u\cdot v<\infty$ and $v>0$

$\therefore\ 0<u<\infty$ ← first limit of integration

$\therefore\ y>0 \longrightarrow u(1-v)>0 \qquad \therefore 1-v>0$ (since $u>0$)

$\therefore\ v<1 \qquad \therefore\ 0<v<1$ ← second limit of integration

$$\therefore \Gamma(\alpha)\cdot\Gamma(\beta) \overset{\text{CVT}}{=} \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} f(u,v)\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

(change of variable theorem)

$$= \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} f(u,v)\; u\cdot du\,dv.$$

since $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ = absolute determinant of Jacobian matrix $= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix}$ since $x = u.v.$, $y = u - uv$

$$= |-\cancel{vu} - u + \cancel{vu}| = |-u| = u.$$

$$= \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} e^{-u}(uv)^{\alpha-1}(u(1-v))^{\beta-1}\,u\;du\,dv$$

$$= \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} e^{-u}\,u^{\alpha-1}v^{\alpha-1}u^{\beta-1}(1-v)^{\beta-1}\,u^{1}\,du\,dv$$

$$= \left[\int_{u=0}^{u=\infty} e^{-u}\,u^{\alpha-1+\beta-1+1}\,du\right]\cdot\left[\int_{v=0}^{v=1} v^{\alpha-1}(1-v)^{\beta-1}\,dv\right]$$

$$= \left[\int_{u=0}^{u=\infty} e^{-u}\,u^{(\alpha+\beta)-1}\,du\right]\cdot\left[\int_{v=0}^{v=1} v^{\alpha-1}(1-v)^{\beta-1}\,dv\right]$$

$$= \Gamma(\alpha+\beta)\cdot B(\alpha,\beta)$$

$$\therefore B(\alpha,\beta) = \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

QED